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The Total Mean Curvature and its some Applications

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Abstract
This study aimed to recognizing the curves and total mean curvature and its some applications. The problem of study is concreted to answer the following question: What are the total mean curvature and its some applications in other fields? The importance of this study is dealing with the study of the total mean curvature through the branches of mathematics such as differential geometry, vectors analysis and how to find the total mean curvature simplest ways. The study followed the historical, mathematical, analytical method and attained the following results: Through calculating the total mean curvature by using derivation method and Hess Matrix, it is found that derivation method is better than Hess Matrix method for its simplicity and the total mean curvature is a numerical value.
Keywords: Applications, Curvature, Total Mean Curvature**.**

1. Introduction

The mean curvature of a discrete M is support along the edges . If e is an edge. And **ecDc** start $(e) = T_1 \cup T_2$ we find

$$H_e = \int\int_D H\, dA = \oint \eta\, ds = e \times v_1 - e \times v_2 = J_1 e - J_2 e$$

Here is v_i is the normal vector to the triangle T_i and J_i is rotation by 90° in the plane of that triangle. Note that$|He|$.

$$|He| = 2\,|e| \sin\frac{\theta_\circ}{2}$$

Where θ_e is the exterior dihedral angle along the edge defined by

$$\cos\theta_e = v_1 . v_2$$

But this discrete mean curvature can be out around the vertices we set.

$$2H_P = \sum_{e \ni P} He = \iint_{star\,(P)} H\, dA = \oint_{link\,(P)} \eta\, dS$$

The area of the discrete surface is a function of the vertex position if we vary only one vertex, we find that

$$\nabla P\ area\ (M) = -H_P \quad \text{[Sullivan, 2006]}$$

2. The Concept of Curve

The point set $r = \{x(t), y(t)\}, a \le t \le b,$ defining as moth arc of a curve when the coordinate functions $x(t), y(t)$ are continuous on their interval of definition $a \le t \le b$.

And when the first derivatives of $x(t)$ and $y(t)$ exist arc continuous and arc -non – zero on $a \leq t \leq b$ the equations.

$$x = x(t), y = y(t), a \leq t \leq b$$

Where $x(t)$ and $y(t)$ define the coordinates of the position vector which determines the smooth arc of a curve, arc called the parametric representation or parametric form of this arc while

$$r = r(t) = \{x(t), y(t)\}, \quad a \leq t \leq b$$

Is called the vector form of this arc. [Mitrinovic, Ulcar, 1969]

Plane Curves For $n = 2$ every regular curve is afferent curve provided it is twice continuously differentiable?

The tangent vector is $e_1 = \grave{c}$ the normal vector is e_2, which if the orientation is positive is the rotation by an angle of $\frac{\pi}{2}$ to the left of the vector e_1 form:

$$0 = \langle \grave{c}, \grave{c} \rangle \grave{=} 2 \langle \grave{c}, \grave{\grave{c}} \rangle = 2\langle e_1, \grave{\grave{c}} \rangle$$

It follows that $\grave{\grave{c}}$ and e_2 arc linearly dependent hence $\grave{\grave{c}} = ke_2$ with some function k.

This function k is said to the (oriented) curvature of c. It is sign indicates in which direction the curve. [Sullivan, 2006]

Is rotating. Here $k > 0$ indicates that the tangent goes to the left, while $k < 0$ indicates that it rotates the right. At an inflection point one has $k = 0$ and the direction of the tangent is stationary. One has the following equations for the derivatives, in which the second follows from the first, since e_2 and e_1 differ by a rotation of $\frac{\pi}{2}$.

$$\grave{e}_1 = \grave{\grave{c}} = k\, e_2$$
$$\grave{e}_2 = -k\, e_1$$

Or using matrix notation

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

Note that the matrix on the right is skew symmetric, which follows already from the relation.

$$0 = \langle e_1, e_2 \rangle = \langle \grave{e}_1, e_2 \rangle + \langle \grave{e}_2, e_1 \rangle$$

These equations are also called the Frenet Equation. [Sullivan, 2006]

Theorem: Plane Curves with Constant Curvature

A regular curve in IR^2 has constant curvature k if and only if it is part of a circle of radius $\frac{1}{|k|}$ $(if\ k \neq 0)$ or a line segment$(if\ k = 0)$.

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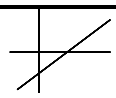
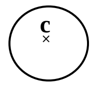
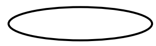
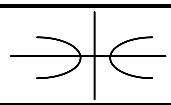
Proof

The proof follows directly from the Frenet Equations. Assume first that $k(s_\circ) \neq 0$ obviously the expression $c(s) + \frac{1}{k(s_\circ)} e_2(s)$ is constants if and only if $c(s)$ is part of a circle of radius $\left|\frac{1}{k(s_\circ)}\right|$, since the difference vector has constant length $\left|\frac{1}{k(s_\circ)}\right|$.

This is equivalent to $k = k(s_\circ)$ every where because $\grave{c} + \frac{1}{k(s_\circ)} \grave{e}_2 = e_1 - \frac{1}{k(s_\circ)} k\, e_1$ the fact that $k = 0$ only holds for line segment is trivial, because $\grave{e}_2 = -k\, e_1$. [Sullivan, 2006]

Example 1.

Name	Implicit equation	Parametric equation	As function	Graph
Straight Line	$ax + by = c$	$(x_\circ + \propto t, y_\circ + \beta t)$	$y = mx + c$	
Circle	$x^2 + y^2 = r^2$	$(r\cos t, r\sin t)$		
Parabola	$y - x^2 = 0$	(t, t^2)	$y = x^2$	
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$(a\cos t, b\sinh t)$		
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$(a\cosh t, b\sin t)$		

Example 2.

i) $\mathbf{c(t) = (at, bt)}$ a line in standard parameterization. Since $c^{\cdot}(t) = (a, b)$ the parameter **is** the arc length if and only if $a^2 + b^2 = 1$ the parameterization $c(t) = (at^3, bt^3)$ describes exactly the same line it is not regular for $t = 0$.

ii) $c(t) = \frac{1}{2}(\cos 2t, \sin 2t)$ a circle of radius $\frac{1}{2}$. Since of course $c^{\cdot}(t) = (-\sin 2t, \cos 2t)$ one has $\left|\left|c^{\cdot}\right|\right| = 1$. Hence it is the arc length i.e, $t = s$. [Sullivan, 2006]

Example 3.

Find sketch the curves with, the parametric from

i) $x = t,\ y = t^2$ ii) $x = t^2, y = t^4$

iii) $x = a\cos t, y = b\sin t$

iv) $x = \cos t, y = \sin^2 t$

v) $x = e^t, y = e^{-t}$

vi) $x = \sqrt{1-t}, y = \frac{1}{t}$

Solution

i) $y = x^2$ *ii)* $y = x^2, x \geq 0$

iii) $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ iv) $y = 1 - x^2, -1 \leq x \leq 1$

v) $xy = 1, x > 0$

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vi) $y = (1 - x^2)^{-1}, x \neq 1$ [Sullivan, 2006]

3. Regular Curves

Many of the basic notions concerning space curves are motivated by the same considerations that were advanced for plane curves. Consequently, there are introduced here rapidly and with little comment. A regular curve is the locus defined by vector.

$$u(t) = \left(u_1(t), (u_2(t), u_3(t)\right), a \leq t \leq b \qquad (4.1)$$

Such that the functions $u(t)$have continuous second or third derivatives (depending on individual circumstances) and such that the derivative $u`(t)$is not zero.

$$u`(t) = (u`_1, u`_2, u`_3) \neq 0 \qquad (4.2)$$

The latter condition ensures that the mapping for the interval into the u_1, u_2, u_3 space is locally one to one; the proof of this fact can be carried out in the same fashion as it was for plane curves. [Stoker, 1969]

Length of Curve

The length $\mathrm{s(t)}$ of the curve extending from an initial point $\mathrm{t_0}$ to a variable point t is defined by. [Stoker, 1969]

$$s(t) = \int_{t_0}^{t} \sqrt{u`(t).u`(t)}\ dt$$

$$= \int_{t_0}^{t} \sqrt{u`^2_1 + u`^2_2 + u`^2_3}\ dt \qquad \text{(Opera, 2007)}$$

Just as with plane curves, the arc length, may always be chosen as parameter since $\frac{ds}{dt} \neq 0$ regularity requires that $u`(t) \neq 0$ hold. In case the arc length is seen from (Opera, 2007) that $|u`(s)| = 1$ conversely, if the relation

$|u`(t)| = 1,$ holds for all $t,$ then $t = s + \text{const}$. [Stoker, 1969]

Examples 4.

Find the lengths of the following space curves :

(i) **[**u(t)={sin^2t, *sint cost*, *cost*} between the points $\mathrm{t} = 0$ and $\mathrm{t} = \mathrm{t_0}$

Solution

$$u`(t) = \{2sint\ cost, -sin^2t + cos^2t, -sint\ \}$$

$$= \{sin\ 2t, cos\ 2t, -sint\ \}$$

$$[u`(t)]^2 = \{sin^2 2t + cos^2 2t + sin^2 t\ \}$$

$$= 1 + sin^2 t$$

$$\sqrt{u`^2(t)} = \sqrt{1 + sin^2 t}$$

$$\because S = \int_0^{t_0} \sqrt{1 + sin^2 t}\ \ dt$$

Let $u = 1 + sin^2 t \ \Rightarrow \frac{du}{dt} = 2sint\ cost = sin\ 2t$

$$dt = \frac{du}{sin\ 2t}$$

When $t = 0 \Rightarrow u = 1 + sin^2 0 = 1 \Rightarrow u = 1$

When $t = t_0 \Rightarrow u = 1 + sin^2 t_0$

$$s = \int_1^{1+sin^2 t_0} u^{\frac{1}{2}} \frac{du}{sin\ 2t}$$

$$= \frac{1}{sin\ 2t} \int_1^{1+sin^2 t_0} u^{\frac{1}{2}}\ du$$

$$= \frac{1}{sin\ 2t} [\tfrac{2}{3} u^{\frac{3}{2}}]_1^{1+sin^2 t_0}$$

$$= \frac{1}{sin\ 2t} [\tfrac{2}{3} (1 + sin^2 t_0)^{\frac{3}{2}} - \tfrac{3}{2}]$$

$$= \frac{2}{3sin\ 2t} [(1 + sin^2 t_0)^{\frac{3}{2}} - 1\ \ ]\ ]$$

(ii) *u(t)* =($rcost, rsint, 0$) for $0 \le t \le 2\pi$

Solution

$$s = \int_{t_0}^{t} \sqrt{u_1^{`2} + u_2^{`2} + u_3^{`2}}\ .dt$$

$$s = \int_0^{2\pi} \sqrt{r^2 sin^2 t + r^2 cos^2 t}\ \ dt$$

$$= \int_0^{2\pi} rdt$$

$$= rt|_0^{2\pi} \ = \ 2\pi r$$ [Opera, 2007]

(iii) *u(t)* = $\{a\ cost, a\ sint, bt\}$ for $0 \le t \le t_0$

Solution

$$s = \int_{t_0}^{t} \sqrt{u`_1^2 + u`_2^2 + u`_3^2}\ .dt$$

$$s = \int_{0}^{t_0} \sqrt{a^2 sin^2 t + a^2 cos^2 t + b^2}\ .dt$$

$$= \int_{0}^{t_0} \sqrt{a^2 + b^2}\ .dt\ = \sqrt{a^2 + b^2} t \mid_{0}^{t_0}$$

$$= \sqrt{a^2 + b^2}\ t_0$$ [Mitrinovic, Ulcar, 1969]

Space Curves

For $n = 3$ a regular three times continuously differentiable curve is called a frequent curve if $\ddot{C}\ \neq 0$ everywhere.

The accompanying three frames are then given by

$e_1 = C`$ (tangent vector)

$e_2 = \frac{C``}{\|C``\|}$ (principal normal vector)

$e_3 = e_1 X\ e_2$ (binormal vector)

The function $K = \|C\|$ is called the curvature of C.

By assumption this number is always positive the equations for the derivatives arc

$\grave{e}_{1=}\ \grave{\grave{C}} = Ke_2$

$$\grave{e}_1 = \langle \grave{e}_2, e_1\rangle e_1 + \underbrace{\langle \grave{e_2}, e_2\rangle}_{0} e_2 + \langle \grave{e}_1, e_3\rangle e_3$$

$$= \langle -e_2\,, \grave{e}_1\rangle e_1 + \underbrace{\langle \grave{e}_2, e_3\rangle}_{T} e_3$$

$$= -Ke_1\ + Te_3$$

$$\grave{e}_3 = \langle \grave{e}_3, e_1\rangle e_1 + \langle \grave{e}_3, e_2\,\rangle e_2\ + \underbrace{\langle \grave{e}_3, e_3\rangle}_{0} e_3$$

$$= -\underbrace{\langle e_3, e`_1\rangle}_{0} e_1 - \underbrace{\langle e_3, e`_2\rangle}_{T} e_2$$

$$= -Te_2$$

$$\therefore \grave{e}_3\ = -Te_2$$

The function $T = \langle e`_2, e_3\rangle$ is called the torsion of C. It indicates how the $(e_1, e_2\,)$plane changes along the curve.

These three equations for the derivatives are called the Frenet equations, and in matrix notation they take the following form: [Kuhnel , 2003]

$$\backslash \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & K & 0 \\ -K & 0 & T \\ 0 & -T & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

4. The Total Mean Curvature

The Curvature

Consider the curver $= \mathrm{r(s)}$, where s is its arc length. If the curve is not straight it is important to have a measure of its deviation from a straight line. Such a measure, is called the curvature$K(s)$, is given by the rate at which the tangent changes direction as it moves along the curve. Let $\emptyset(s)$be the angle, measured in the positive sense, between the tangent direction $r`(s)$ at s and direction i. The curvature $K(s)$ of a curve $r(s)$ at the point s is then defined by

$$K(s) = \frac{d\emptyset(s)}{ds} \qquad (1)$$

it follows from (1) that :

$$K = x`y`` - y`x`` \qquad (2)$$

The vector $r``$ is a vector normal to the tangent vector $r`$ such that

$$|K| = |r``| \qquad (3)$$

if $K \neq 0$, it follows that

$$t = r`, \qquad n = \frac{1}{K} r`` \qquad (4)$$

where t and n arc the unit tangent and unit normal to the curve $r(s)$ at s. At every point of a curve, the vectors t and n are equally inclined to i and j, respectively.

The sign of the curvature, which has a geometrical interpretation , is given by (2) if $r``$ is continuous at the point $s_\circ$, then in a neigh bourhood of the point $s_\circ$ of $r = r(s)$ the curve lies on the side of the tangent to which n points when $K(s_\circ) > 0$ and opposite side when $K(s_\circ) < 0$.

As a rule, a curve will be given in parametric form where the parameter t is not the arc length. In this case the curvature is given by [math, 2016]

$$K = \left| \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}} \right|$$

for all curves of the form $y = f(x)$

Example 5.

If $x = 4sin\ t\ , y = cos\ t$

Find the curvature K

Solution

$$\dot{x} = 4\cos t\,, \qquad \ddot{x} = -4\sin t$$

$$y^{\cdot} = -\sin t\ , \qquad y^{\cdot\cdot} = -\cos t$$

$$\because \quad K = \left|\frac{x^{\cdot}y^{\cdot\cdot} - y^{\cdot}x^{\cdot\cdot}}{(x^{\cdot 2} + y^{\cdot 2})^{\frac{3}{2}}}\right|$$

$$K = \left|\frac{(4\cos t)(-\cos t) - (-\sin t)(-4\sin t)}{((4\cos t)^2 + (-\sin t)^2)^{\frac{3}{2}}}\right|$$

$$K = \left|\frac{-4\cos^2 t - 4\ \sin^2\ t}{(16\cos^2 t + \ \sin^2 t)^{\frac{3}{2}}}\right|$$

$$K = \left|\frac{-4\,(\cos^2 t + \ \sin^2\ t)}{(15\cos^2 t + \ \cos^2 t + \sin^2 t)^{\frac{3}{2}}}\right|$$

$$K = \left|\frac{-4}{(15\cos^2 t + 1)^{\frac{3}{2}}}\right| = \frac{4}{1 + 15\cos^2 t}$$

$$\therefore \quad K = \frac{4}{1 + 15\cos^2 t} \quad]$$

Example 6.

Evaluate the radius of curvature at the given point M:

i- $x = t,\ y = t^3, M = (1,1)$

ii- $x = t^2,\ y = t^3, M = (1,1)$

iii- $x = a(\cos t + t\sin t)\,,\ y = a(\sin t - t\cos t)$

Solution

i- $x^{\cdot} = 1,\ x^{\cdot\cdot} = 0, y^{\cdot} = 3t^2,\ y^{\cdot\cdot} = 6t$

$$\because K = \left|\frac{x^{\cdot}y^{\cdot\cdot} - y^{\cdot}x^{\cdot\cdot}}{(x^{\cdot 2} + y^{\cdot 2})^{\frac{3}{2}}}\right| = \left|\frac{(1)(6t) - (3t^2)(0)}{(\,(1)^2 + (3t^2)^2\,)^{\frac{3}{2}}}\right|$$

$$= \left.\frac{6t - 0}{(1 + 9t^4)^{\frac{3}{2}}}\right|_{t=1}$$

$$= \frac{6}{(1+9\,)^{\frac{3}{2}}} = \frac{6}{10\sqrt{10}} = \frac{3}{5\sqrt{10}}$$

ii- $x^{\cdot} = 2t,\ x^{\cdot\cdot} = 2,\ \ y^{\cdot} = 3t^2,\ y^{\cdot\cdot} = 6t$

$$K = \left|\frac{(2t)(6t) - (3t^2)(2)}{((2t)^2 + (3t^2)^2)^{\frac{3}{2}}}\right| = \left|\frac{12t^2 - 6t^2}{(4t^2 + 9t^4)^{\frac{3}{2}}}\right|$$

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$$= \frac{6t^2}{(4t^2+9t^4)^{\frac{3}{2}}}\Bigg|_{t=1}$$

$$= \frac{6}{(4+9)^{\frac{3}{2}}} = \frac{6}{13\sqrt{13}}$$

iii- $\dot{x} = \mathrm{a}(-\sin t + t\cos t + \sin t) = a(t\cos t)$

$\ddot{x} = \mathrm{a}(-t\sin t + \cos t)$

$\dot{y} = \mathrm{a}(\cos t + t\sin t - \cos t) = a(t\sin t)$

$\ddot{y} = a(t\cos t + sint)$

$$K = \left|\frac{a(t\cos t)(a(t\cos t+\sin t) - a(t\sin t)(a(-t\sin t+\cos t)}{((a(t\cos t))^2 + (a(t\sin t))^2)^{\frac{3}{2}}}\right|$$

$$K = \left|\frac{a^2\, t^2\, cos^2\, t \;\; + a^2 \;\; t\, sin\, t\, cos\, t + a^2\, t^2\, sin^2\, t - a^2 t\;\, sin\, t\, cos\, t}{(a^2 t^2\, cos^2\, t \;\; + \;\; a^2 t^2\, sin^2\, t)^{\frac{3}{2}}}\right|$$

$$K = \left|\frac{a^2 t^2\,(\cos^2 t + \;\sin^2 t)}{(a^2 t^2\,(\cos^2 t + \;\sin^2 t)\,)^{\frac{3}{2}}}\right|$$

$$= \left|\frac{a^2 t^2}{(a^2 t^2)^{\frac{3}{2}}}\right| = \frac{a^2 t^2}{(a^2 t^2)\sqrt{a^2 t^2}} = \frac{1}{at}$$ [Lee, 2000]

5. The Mean Curvature

In mathematics, the mean curvature H of a surface is an intrinsic measure of curvature that comes from differential geometry and that locally describes the curvature of an embedded surface in some ambient space such as Euclidean space. [wikipedia, 2016].

The concept was introduced by Sophie Germaine in her work on elasticity theory. It is important in the analysis of minimal surfaces which have mean curvature zero, and in the analysis of physical interfaces between fluids (such as soap films) which by the young-Laplace's equation have constant mean curvature. [wikipedia, 2016]

6. Definition

Let P be a point on the surfaces. Each plane through P containing the normal line to **s** cuts *sin a* (plane) curve. Fixing a choice of unit normal gives a signed curvature to that curve. As the plane is rotated (always containing the normal line) that curvature can vary, and the maximal curvature K_1, and minimal curvature K_2 are known as the principal curvatures of s. The mean curvature at $P \in S$ is then the average of the principal curvatures hence the name.[hotmathwomensfonm,2016]

$$H = \frac{1}{2}(K_1 + K_2)$$ [wikipedia, 2016]

more generally for a hyper surface T the mean curvature is given as

$$H = \frac{1}{n}\sum_{i=1}^{n} Ki$$

Implicit Form of Mean Curvature

The mean curvature of a surface specified by an implicit equation $F(x, y, z) = 0$ can be calculated by using the gradient. [Promoted, 2016]

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$$

Application

If $r_1(t) = (a\cos t, a\, sint)\ and\ r_2(t) = (t^2, t^3)$

are two curves then find the total mean curvature at

$t = 1$

Solution

$$K_1 = \left|\frac{x^{\cdot} y^{\cdot\cdot} - y^{\cdot} x^{\cdot\cdot}}{(x^{\cdot 2} + y^{\cdot 2})^{\frac{3}{2}}}\right|$$

$x^{\cdot} = -a\sin t, \qquad x^{\cdot\cdot} = -a\cos t$

$y^{\cdot} = a\cos t, \qquad y^{\cdot\cdot} = -a\sin t$

$$K_1 = \left|\frac{(-a\ \sin t)(-a\ \sin t) - (a\cos t)(-a\cos t)}{((-a\ \sin t)^2 + (a\ \cos t)^2)^{\frac{3}{2}}}\right| = \left|\frac{a^2\ \sin^2 t + a^2\ \cos^2 t}{(a^2\ \sin^2 t + a^2\ \cos^2 t)^{\frac{3}{2}}}\right|$$

$$= \left|\frac{a^2}{(a^2)^{\frac{3}{2}}}\right| = \left|\frac{a^2}{a^3}\right| = \frac{1}{a} \therefore K_1 = \frac{1}{a}$$

$$K_2 = \left|\frac{x^{\cdot} y^{\cdot\cdot} - y^{\cdot} x^{\cdot\cdot}}{(x^{\cdot 2} + y^{\cdot 2})^{\frac{3}{2}}}\right|$$

$x^{\cdot} = 2t, x^{\cdot\cdot} = 2,\ y^{\cdot} = 3t^2,\ y^{\cdot\cdot} = 6t$

$$K_2 = \left|\frac{(2t)(6t) - (3t^2)(2)}{((2t)^2 + (3t^2)^2)^{\frac{3}{2}}}\right| = \left|\frac{12t^2 - 6t^2}{(4t^2 + 9t^4)^{\frac{3}{2}}}\right| = \frac{6t^2}{(4t^2 + 9t^4)^{\frac{3}{2}}}\Bigg|_{t=1} = \frac{6}{(4+9)^{\frac{3}{2}}} = \frac{6}{(13)^{\frac{3}{2}}} = \frac{6}{13\sqrt{13}}$$

$$\therefore K_2 = \frac{6}{13\sqrt{13}}$$

$$\because H = \frac{1}{2}(K_1 + K_2)$$

$$H = \frac{1}{2}\left(\frac{1}{\text{a}} + \frac{6}{13\sqrt{13}}\right) = \frac{1}{2}\left(\frac{13\sqrt{13} + 6\text{a}}{13\sqrt{13}\text{a}}\right)$$

$$H = \frac{13\sqrt{13} + 6\text{a}}{26\sqrt{13}\,\text{a}} \quad]$$

Hessian Matrix

$$Hess(F) = \begin{bmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial x \partial z} \\ \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial y^2} & \frac{\partial^2 F}{\partial y \partial z} \\ \frac{\partial^2 F}{\partial x \partial z} & \frac{\partial^2 F}{\partial y \partial z} & \frac{\partial^2 F}{\partial z^{\,2}} \end{bmatrix}$$

The mean curvature is given by :

$$H = \frac{\nabla F\ Hess(F) \nabla F^T - |\nabla F|^2\ Trace\ (Hess(F))}{2|\nabla F|^3}$$

Another form is as the divergence of the unit normal. A unit normal is given by $\frac{\nabla F}{|\nabla F|}$ and the mean curvature is

$$H = \frac{-1}{2}\nabla\left(\frac{\nabla F}{|\nabla F|}\right) \qquad [\text{wikipedia}, 2016]$$

Application

[If $F = x^3 + 2xy + yz + y^3 + 2xyz + z^2$

find the total mean curvature at the point $(1, -1, 1)$

Solution

$$Hess(F) = \begin{bmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial x \partial z} \\ \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial y^2} & \frac{\partial^2 F}{\partial y \partial z} \\ \frac{\partial^2 F}{\partial x \partial z} & \frac{\partial^2 F}{\partial y \partial z} & \frac{\partial^2 F}{\partial z^{\,2}} \end{bmatrix}$$

$$\frac{\partial F}{\partial x} = 3x^2 + 2y + 2yz$$

$$\frac{\partial F}{\partial x^2} = 6x, \qquad \frac{\partial^2 F}{\partial x \partial y} = 2 + 2z, \qquad \frac{\partial^2 F}{\partial x \partial z} = 2y$$

$$\frac{\partial F}{\partial y} = 2x + z + 3y^2 + 2xz, \qquad \frac{\partial^2 F}{\partial y^2} = 6y$$

$$\frac{\partial^2 F}{\partial y \partial z} = 1 + 2x, \qquad \frac{\partial F}{\partial z} = y + 2xy + 2z$$

$$\frac{\partial^2 F}{\partial z^2} = 2$$

$$Hess(F) = \left.\begin{bmatrix} 6x & 2 + 2z & 2y \\ 2 + 2z & 6y & 1 + 2x \\ 2y & 1 + 2x & 2 \end{bmatrix}\right|_{(1,-1,1)}$$

$$Hess(F) = \begin{bmatrix} 6 & 4 & -2 \\ 4 & -6 & 3 \\ -2 & 3 & 2 \end{bmatrix}$$

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$$

$$\nabla F = (3x^2 + 2y + 2yz, 2x + z + 3y^2 + 2xz, y + 2xy + 2z\,)\,|_{(1,-1,1)}$$

$$\nabla F = (-1, 8, -1)$$

$$|\nabla F|^2 = 1 + 64 + 1 = 66$$

$$|\nabla F|^3 = |\nabla F|^2 |\nabla F| = 66\sqrt{66}$$

$$Hess(F)\nabla F^T = \begin{bmatrix} 6 & 4 & -2 \\ 4 & -6 & 3 \\ -2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 6 & 4 & -2 \\ 4 & -6 & 3 \\ -2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 56 & -6 & -4 \\ -6 & 61 & -20 \\ -4 & -20 & 17 \end{bmatrix}$$

$$Hess(F) = 6(-12 - 9) - 4(8 + 6) - 2(12 - 12) = 6(-21) - 4(14) - 0$$

$$= -126 - 56 = -182$$

$$Hess(F) = -182$$

$$Hess(F)\nabla F^T = 56(637) + 6(-182) - 4(364)$$

$$= 35672 - 1092 - 1456 = 33124 \therefore \quad Hess(F)\nabla F^T = 33124$$

$$\because H = \frac{\nabla F Hess(F)\nabla F^T - |\nabla F|^2\, Trace\ Hess(F)}{2|\nabla F|^3}$$

$$H = \frac{\sqrt{66}\ (33124) - 66(182)}{2\left(66\sqrt{66}\,\right)} = \frac{\sqrt{66}\,(33124) - \ 12012}{132\sqrt{66}} = \frac{8.12(33124) - 12012}{(132)(8.12)}$$

$$= \frac{268966.88 - 12012}{1071.84} = \frac{256954.88}{1071.84} = 239.73 \therefore H = 239.73]$$

Conclusion

Through calculating the total mean curvature by using derivation method and Hess matrix, we found that derivation method is better than Hess matrix method for its simplicity and the total mean curvature is a numerical value. Through using derivation method and Hess matrix for calculating the total mean curvature, we found that Hess Matrix method takes longer time, while derivation method is more simple from Hess Matrix method, so we recommend to future researchers finding another methods better than these two methods and also we recommend to future researchers find advanced applications about the total mean curvature. Finally we want to say that derivation method is the best than Hess matrix method for its simplicity.

References

D.S Mitrinovic and J. Ulcar, Differential Geometry, tutorial text No. 5, Belegrade, Yogoslavia, 1969.

Jeffrey M. Lee, Differential Geometry Analysis and Physics, first edition, 2000.

John Opera, Differential Geometry and its Applications, second edition, United States of America ,2007.

James J. Stoker, Differential Geometry, first edition, United States of America, 1969.

John M. Sullivan, Differential Geometry, first edition, 2006.

Woiffang Kuhnel, Differential Geometry Curves Surfaces Manifolds, second edition, Wieshaden, 2003.

hotmath.com. womensfonm.comglamfamily, June, 2016.

Promoted stories from around the web, June, 2016.

www.math.ubc.ca/nfeldman/m200/vectorppties, June, 2016.

En.wikipedia.org, June, 2016.