



Series of Weighted Restriction Functions for Space Curves

Yasir Mohmed Alamen¹ and Abdurahim Mohammed Ebraheem²

^{1,2} University of Kassala . Faculty of Education, Department of Mathematics, Sudan
 Crspending Auther Email; aboibrahem2009@yahoo.com

Abstract

Consider a nondegenerate C^n curve $\gamma(t)$ in \mathbb{R}^n , $n \geq 2$, such as the curve $\gamma_0(t) = (t, t^2, \dots, t^n)$, $t \in I$, where I is an interval in \mathbb{R}^n . We first show a weighted Fourier restriction theorem for such curves, with a weight in a Wiener amalgam space, for the full range of exponents p, q , when I is a finite interval. Next, we obtain a generalization of this result to some related oscillatory integral operators. In particular, the results suggest that this is a quite general phenomenon which occurs, for instance, when the associated oscillatory integral operator acts on functions f with a fixed compact support. We show an analogue, for the Fourier extension operator (i.e. the adjoint of the Fourier restriction operator), of the two-weight norm inequality of B. Mockenhaupt for the Fourier transform. Here I may be either finite or infinite. These results extend two results of J. Lakey on the plane to higher dimensions.

1. Restriction with weighted function in an amalgam space

Let $n = 2 + \epsilon$, $\epsilon \geq 0$ and consider a nondegenerated C^n curve $\gamma(t)$ in \mathbb{R}^n , with t in a finite interval I . Namely, we assume that

$\left| \det \left(\gamma'_x(u_1), \dots, \gamma'_x(u_n) \right) \right| \geq c > 0$ on I . A prototypical example is $\gamma_0(t) = (t, t^2, \dots, t^n)$.

We prove a weighted Fourier restriction theorem for such curves, with a weight function in a Wiener amalgam space $W(L^1, \ell^\infty)$. Recall that the norm on the amalgam space $W(L^p, \ell^q)(\mathbb{R}^n)$ is defined by

$$\|f\|_{W(L^p, \ell^q)} = \left(\sum_{k \in \mathbb{Z}^n} \left(\int_{Q_k} |f(x)|^p dx \right)^{q/p} \right)^{1/q}$$

if $1 \leq p, q < \infty$ (and with the usual L^∞ or ℓ^∞ modification when p or q is infinite). Here \mathbb{Z}^n is an integer lattice (i.e. the set of all n -tuples of integers) and $Q_k = Q + k$ is a translate of the unit cube $= [0, 1]^n$. Note that $\|f\|_{W(L^p, \ell^q)}$ is just the mixed norm:

$$\sum \|f \chi_{Q_k}\|_{\ell^q(L^p)} = \sum \left\| \|f \chi_{Q_k}\|_{L^p(\mathbb{R}^n)} \right\|_{\ell^q(\mathbb{Z}^n)}$$

See [9,10] for more details on amalgam spaces.

The result may be stated as follows. As usual, $p' = p/(p-1)$ is the Hölder conjugate exponent of p , $1 \leq p \leq \infty$, and \hat{f} is the Fourier transform of complex function f on \mathbb{R}^n . We adopt the usual convention that C denotes a constant which is uniform in a suitable sense, but whose value may not be the same at each occurrence.

Theorem (1.1)[14]: Let $\gamma(t)$ be a nondegenerate C^n curve in \mathbb{R}^n , $t \in I$ where I is a finite interval. Let

$$1 \leq p < ((2 + \epsilon)^2 + (2 + \epsilon) + 2) / ((2 + \epsilon)^2 + (2 + \epsilon)),$$

and

$$1/p + [2 / ((2 + \epsilon)^2 + (2 + \epsilon))] / q \geq 1.$$

Suppose that v is a nonnegative (measurable) function such that $v^{1-p'} \in W(L^1, \ell^\infty)$. Then there exists a constant C , independent of f and v , such that



$$\left(\int_I |\hat{f}(\gamma(t))|^q dt \right)^{1/q} \leq C \|v^{1-p'}\|_{W(L^1, \ell^\infty)}^{1/p'} \left(\int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{1/p} \quad (1.1)$$

Here, C depends only on γ, n and I .

Notice that (1.1) is invariant under the dilation $v \mapsto Av$ for any constant $A > 0$, since

$$\|v^{1-p'}\|_{W(L^1, \ell^\infty)}^{1/p'} = \|v^{-1/p}\|_{W(L^{p'}, \ell^\infty)}.$$

The range of p, q given above is optimal at least for the model case $\gamma_0(t) = (t, t^2, \dots, t^n)$.

Namely, for (1.1) to hold the conditions

$$(i) 1 \leq p < ((2 + \epsilon)^2 + (4 + \epsilon)) / ((2 + \epsilon)^2 + (2 + \epsilon)) \text{ and}$$

$$(ii) 1/p + [2 / ((2 + \epsilon)^2 + (2 + \epsilon))] / q \geq 1$$

are necessary. In fact this just follows from the known necessary conditions in the unweighted case $v \equiv 1$. For the necessity of (i) in the case of γ_0 , see [1, 2]. The necessary condition (ii) is valid when I is a finite interval. This follows from a well-known homogeneity argument, and this has been shown to be valid in a great generality (see, e.g., [3]). (See Fig. 1. Here, $A = A(n)$ is the point

$((n^2 + n) / (n^2 + n + 2), (n^2 + n) / (n^2 + n + 2))$, and so $A = (6/7, 6/7)$ when $n = 3$).

The classical case (i.e. the unweighted case, $v \equiv 1$) of Theorem (1.1) was proved by Drury [5] (see also [3, 4, 6-8], and references contained there). In the case $n = 2$, Lakey [11] (Theorem 4.1 and Corollary 4.3) showed the above result for the full range of exponents p, q . He also obtained a partial result for $n \geq 3$ (see Remark 1.5 below).

Remark (1.2)[14]: Examples. Fix

$$1 < p < ((2 + \epsilon)^2 + (4 + \epsilon)) / ((2 + \epsilon)^2 + (2 + \epsilon)),$$

and

$$1/(p' - 1) - \epsilon < a < 1/(p' - 1),$$

for some small $\epsilon > 0$. Let $v(x) = \prod_{j=1}^n |\sin(2\pi x_j)|^a$, $x \in \mathbb{R}^n$, or let $v(x) = |x|^{na}$ for $x \in Q = [0, 1]^n$, and extend it to be periodic. Then $v^{-(p'-1)} \in W(L^1, \ell^\infty)$, and there exist functions f ,

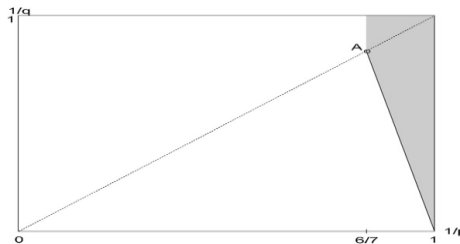


Fig. 1. The range of $(1/p, 1/q)$ in Theorem 1.1 for $n = 3$.

which belong to $L_v^p(\mathbb{R}^n) = L^p(\mathbb{R}^n, v dx)$, but not to $L^p(\mathbb{R}^n)$. For such f , (1.1) implies that the restriction of the Fourier transform \hat{f} to the curve γ is a well-defined function in $L^q(I, dt)$.

Remark (1.3): Since $W(L^r, \ell^s) \subset W(L^1, \ell^\infty)$ for $r, s \in [1, \infty]$, it follows that the condition $V = v^{1-p} \in W(L^r, \ell^s)$ is the weakest when $r = 1, s = \infty$.

The proof relies on the following basic fact on amalgam spaces. It is an expression of the fact that the Fourier transform of a function of compact support is nearly constant on unit balls of an arbitrary center. In fact, this appears to be related to a quite general phenomenon about oscillatory integrals (see Section 2, especially (2.4)). This property is reflected in the somewhat curious fact that the compactness of the curve γ makes a weight v with $v^{1-p'} \in W(L^1, \ell^\infty)$ behave as if it were the reciprocal of a bounded function.



Corollary(1.4) [14]: Let K be a compact set in \mathbb{R}^n and $f \in L^1(\mathbb{R}^n)$ with support K . Let $\hat{f} \in L^r(\mathbb{R}^n)$ for some $r \in [1, \infty)$. Then $\hat{f} \in W(L^\infty, \ell^r)$ and there exists a constant A , depending only on n, K and r , such that

$$\sum \|\hat{f}\|_{L^r(\mathbb{R}^n)} \leq \sum \|\hat{f}\|_{W(L^\infty, \ell^r)} \leq A \sum \|\hat{f}\|_{L^r(\mathbb{R}^n)} \quad (1.2)$$

Observe that an application of Hölder's inequality to (27) and the equality

$$\|\hat{f}\|_{W(L^r, \ell^r)} = A \|\hat{f}\|_{L^r(\mathbb{R}^n)}$$

show that if $r \leq s \leq \infty$, then

$$\sum \|\hat{f}\|_{L^r(\mathbb{R}^n)} \leq \sum \|\hat{f}\|_{W(L^s, \ell^r)} \leq A^{1-r/s} \sum \|\hat{f}\|_{L^r(\mathbb{R}^n)}.$$

Proof: The proof of (1.2) closely follows that given by Holland [10] for the case $n = 1$. We give an outline here for the convenience of the reader. The first inequality is obvious. To show the second inequality, choose a smooth function g with compact support such that $g = 1$ on K and note that

$\hat{f}(x) = \int_K e^{-2\pi i x \cdot \xi} f(\xi) d\xi = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(\xi) g(\xi) d\xi = \int \hat{g}(y) \hat{f}(x - y) dy$. By Hölder's inequality

$$\begin{aligned} |\hat{f}(x)|^r &\leq \left(\int |\hat{g}(y)|^{1/r} |\hat{f}(x - y)| \cdot |\hat{g}(y)|^{1/r'} dy \right)^r \\ &\leq \int |\hat{g}(y)| \cdot |\hat{f}(x - y)|^r dy \cdot \|\hat{g}\|_1^{r/r'} = \int |\hat{g}(x - y)| \cdot |\hat{f}(y)|^r dy \cdot \|\hat{g}\|_1^{r/r'}. \end{aligned}$$

Taking a supremum over $x \in Q_k$ and summing over $k \in \mathbb{Z}^n$ on both sides, we get

$$\sum \|\hat{f}\|_{W(L^\infty, \ell^r)} \leq A \sum \|\hat{f}\|_{L^r(\mathbb{R}^n)}$$

Where $A = 2^{(2+\epsilon)/r} \|\hat{g}\|_{W(L^\infty, \ell^1)}^{1/r} \|\hat{g}\|_1^{1/r'}$. This is because, for each fixed y ,

$$\sum_{k \in \mathbb{Z}^n} \sup_{x \in Q_k} |\hat{g}(x - y)| \leq \sum_{k \in \mathbb{Z}^n} \sup_{x \in Q_k} |\hat{g}(x)| = 2^{(2+\epsilon)} \|\hat{g}\|_{W(L^\infty, \ell^1)}.$$

This proves (1.2).

Proof of theorem (1.1): It suffices to show the dual estimate

$$\sum \|v^{-1/p} T f\|_{L^{p'}(\mathbb{R}^n)} \leq C \sum \|v^{1-p'}\|_{W(L^1, \ell^\infty)}^{1/p'} \|f\|_{L^{q'}(I, dt)} \quad (1.3)$$

where T is the Fourier extension operator (i.e. the adjoint of the Fourier restriction operator):

$$(Tf)(\xi) = \int_I e^{2\pi i \xi \cdot \gamma(t)} f(t) dt.$$

As in [5] and [3] we write

$$(Tf)(\xi)^{(2+\epsilon)} = \int_{I^n} e^{2\pi i \xi \cdot [\gamma(t_1) + \dots + \gamma(t_n)]} f_1(t_1) \dots f_n(t_n) dt_1 \dots dt_n = \hat{F}(\xi)$$

by making the change of variables $(t_1, \dots, t_n) \mapsto (x_1, \dots, x_n) = (\gamma(t_1) + \dots + \gamma(t_n))$. The Jacobian of this transformation is a constant multiple of the Vandermonde determinant. Thus, we have written $(Tf)(\xi)^n$ as the Fourier transform of a compactly supported function F . We need to estimate

$$\sum \|v^{-1/p} T f\|_{p'}^{p'} = \sum \left(\int V(\xi) |\hat{F}(\xi)|^{p'/(2+\epsilon)} d\xi \right)$$

where $V = v^{1-p'}$. This integral is bounded by

$$\|V\|_{W(L^1, \ell^\infty)} \|\hat{F}\|_{W(L^\infty, \ell^1)}^{p'/(2+\epsilon)} = \|V\|_{W(L^1, \ell^\infty)} \|\hat{F}\|_{W(L^\infty, \ell^{p'/(2+\epsilon)})}^{p'/(2+\epsilon)}.$$



Since F is compactly supported, by Lemma (1.4) the last expression is equivalent to

$$\begin{aligned} & \|V\|_{W(L^1, \ell^\infty)} \left\| (Tf)^{\|V\|_{W(L^1, \ell^\infty)} \|\hat{F}^{p'/(2+\epsilon)}\|_{W(L^\infty, \ell^1)}} \right\|_{L^{\frac{p'}{\|V\|_{W(L^1, \ell^\infty)} \|\hat{F}^{p'/(2+\epsilon)}\|_{W(L^\infty, \ell^1)}}} \\ &= \|V\|_{W(L^1, \ell^\infty)} \|Tf\|_{L^{p'}(\mathbb{R}^n)}. \end{aligned}$$

Thus, we have effectively reduced the problem to the case $v \equiv 1$, i.e. a restriction estimate without a weight. By the results in [5, 6] (see also [3, 4] for generalizations), we have

$$\sum \|Tf\|_{L^{p'}(\mathbb{R}^n)} \leq C \sum \|f\|_{L^{q'}(I, dt)} \quad (1.4)$$

for $1 \leq p < ((2 + \epsilon)^2 + (4 + \epsilon)) / ((2 + \epsilon)^2 + (2 + \epsilon))$,

$1/p + [2 / ((2 + \epsilon)^2 + (2 + \epsilon))] / q = 1$. This finishes the proof of (1.3).and

Remark (1.5)[14]: We would like to explain briefly the main difference between the present argument and the argument in [11].

The restriction estimate (1.1) was proved in [11] for the points $(1/p, 1/q)$ on the segment given by

$$\begin{aligned} & 1/p + [2 / ((2 + \epsilon)^2 + (2 + \epsilon))] / q = 1, 1 \leq p < p_1 \\ & := \frac{((2 + \epsilon)^2 + 2(2 + \epsilon))}{((2 + \epsilon)^2 + (2 + 2\epsilon))}, \end{aligned}$$

With $p_1 < p_\infty := ((2 + \epsilon)^2 + (4 + \epsilon)) / ((2 + \epsilon)^2 + (2 + \epsilon))$.

This result was obtained by using the Hausdorff-Young inequality for amalgam spaces, and it corresponds to the present result obtained by the first iterative step (as described below). To be more precise, the relevant steps in [11] are

$$\sum \|\hat{F}\|_{W(L^\infty, \ell^{p'/(2+\epsilon)})} \leq C \sum \|F\|_{W(L^{(p'/(2+\epsilon))', \ell^1)} \leq C \sum \|F\|_{L^{(p'/(2+\epsilon))'}(\mathbb{R}^n)}.$$

The first inequality follows from the Hausdorff-Young inequality of Holland [10] (see also [9]), and the second inequality is a consequences of the compactness of the support of F . Finally, the last norm is estimated by using essentially the weak type behavior of a power of the Vandermonde determinant, which arose as the Jacobian of the change of variables introduced in the beginning of the proof. On the other hand, proof uses the compactness of the support of F slightly differently. Namely, we use it to replace the norm $\|\hat{F}\|_{W(L^\infty, \ell^{p'/(2+\epsilon)})}$ by the equivalent expression

$$\|\hat{F}\|_{L^{p'/(2+\epsilon)}(\mathbb{R}^n)} = \|(Tf)^n\|_{L^{p'/(2+\epsilon)}(\mathbb{R}^n)}$$

using Lemma (1.4). This enables us to use a result (without a weight function) obtained by an inductive argument, which was originally due to Drury [5] (see [3] for an easy version of this argument. In other words, in order to estimate $\|\hat{F}\|_{L^{p'/(2+\epsilon)}(\mathbb{R}^n)}$ or $\|(Tf)^n\|_{L^{p'/(2+\epsilon)}(\mathbb{R}^n)}$, we avoid direct use of the Hausdorff-Young inequality. Instead, we use the Plancherel theorem and a restriction estimate (1.4), established in a previous step (for $1 \leq p < p_j$). (Induction starts with the trivial estimate for $p_0 = 1, q_0 = \infty$). Interpolating these two estimates yield (1.4) for a wider range (for $1 \leq p < p_{j+1}$). Iterating this process gives (1.4), hence (1.1), on the entire half-open segment given by $1/p + [2 / ((2 + \epsilon)^2 + n)] / q = 1, 1 \leq p < p_\infty := ((2 + \epsilon)^2 + (4 + \epsilon)) / ((2 + \epsilon)^2 + (2 + \epsilon))$, since p_j is an increasing sequence converging to p_∞ .

2. Extension to oscillatory integral operators

We can generalize Theorem (1.1) to a class of oscillatory integral operators, related to restriction estimates, as was considered in [3]. The following lemma is obvious.



Lemma (2.1)[14]: Let Q be a cube in \mathbb{R}^n of side length 1. Given measurable functions f and g on Q , suppose that

$$\sum_{x \in Q} \sup |f(x)| \leq C \sum_{x \in Q} \inf |g(y)|, \quad (2.1)$$

with C independent of Q . Then

$$\sum \|f\|_{W(L^\infty, \ell^r)} \leq A \sum \|g\|_{L^r(\mathbb{R}^n)}.$$

Let $a(x)$ and $b(\xi)$ be smooth functions supported in the unit cubes $Q(0,1) := Q^n(0,1) \subset \mathbb{R}^n$ and $Q^m(0,1) \subset \mathbb{R}^m$, respectively. For $\lambda \gg 1$, let us set

$$(T_\lambda f)(x) = a(x/\lambda) \int_{\mathbb{R}^m} e^{i\lambda\phi(x/\lambda, \xi)} b(\xi) f(\xi) d\xi,$$

where ϕ is a smooth function on $Q(0,1) \times Q^m(0,1)$. Let $\|T_\lambda\|_{s \rightarrow r}$ denote the $L^s(\mathbb{R}^m) - L^r(\mathbb{R}^n)$ operator norm of T_λ .

Theorem (2.2)[14]: Let T_λ be given as above. If $v^{1-p'} \in W(L^1, \ell^\infty)(\mathbb{R}^n)$, then

$$\sum \|v^{-1/p} T_\lambda f\|_{L^{p'}(\mathbb{R}^n)} \leq C \sum \|v^{-1/p}\|_{W(L^{p'}, \ell^\infty)} \|T_\lambda\|_{q' \rightarrow p'} \|f\|_{L^{q'}(\mathbb{R}^m)} \quad (2.2)$$

uniformly in $\lambda \gg 1$.

Now consider the special case $m = 1$ of T_λ :

$$(S_\lambda f)(x) = a(x/\lambda) \int_{\mathbb{R}} e^{i\lambda\phi(x/\lambda, t)} b(t) f(t) dt$$

where the nondegeneracy condition

$$\det(\partial_t(\nabla_x \phi), \partial_t^2(\nabla_x \phi), \dots, \partial_t^n(\nabla_x \phi))(x, t) \neq 0$$

is satisfied on the support of the cutoff function $a(x)b(t)$. It was shown in [3] that

$$\|S_\lambda\|_{q' \rightarrow p'} \leq C_{p,q} \quad (2.3)$$

uniformly in $\lambda \gg 1$, if

$$1 \leq p < ((2 + \epsilon)^2 + (4 + \epsilon)) / ((2 + \epsilon)^2 + (2 + \epsilon)),$$

$$1/p + [2 / ((2 + \epsilon)^2 + (2 + \epsilon))] / q \geq 1. \text{ and}$$

(Here the decay factor $\lambda^{-(n+\epsilon)}$ that was present in the operator norm has been absorbed by the rescaling $x \mapsto x/\lambda$). Therefore, we have

Corollary (2.3)[14]: Let S_λ be given as above. If $v^{1-p'} \in W(L^1, \ell^\infty)(\mathbb{R}^n)$, then

$$\sum \|v^{-1/p} S_\lambda f\|_{L^{p'}(\mathbb{R}^n)} \leq C \sum \|v^{-1/p}\|_{W(L^{p'}, \ell^\infty)} \|f\|_{L^{q'}(\mathbb{R}^m)}, \text{ uniformly in } \lambda \gg 1, \text{ if}$$

$$1 \leq p < ((2 + \epsilon)^2 + (4 + \epsilon)) / ((2 + \epsilon)^2 + (2 + \epsilon)) \text{ and}$$

$$1/p + [2 / ((2 + \epsilon)^2 + (2 + \epsilon))] / q \geq 1.$$

Proof of Theorem (2.2): We claim that for any on $Q = Q(x_0, 1) \subset \lambda Q(0,1)$, we have for any $x, y \in Q$ and $N > 0$,

$$|T_\lambda f(x)| \leq C_N \sum_{l \in \mathbb{Z}^n} (1 + |l|)^{-N} |T_\lambda(f e^{-2\pi i \xi \cdot l})(y)|, \quad (2.4)$$

with C_N independent of Q . Thus, we have verified the hypothesis of Lemma (2.1) for some pair of functions. Therefore, we may apply Lemma (2.1). Then we write $(1 + |l|)^{-N} = (1 + |l|)^{-N/r'} (1 + |l|)^{-N/r}$, and apply Hölder's inequality for series to obtain

$$\sum_{k \in \mathbb{Z}^n} \sup_{x \in Q} |T_\lambda f(x)|^r \leq C \sum_{l \in \mathbb{Z}^n} (1 + |l|)^{-N} \int_{\mathbb{R}^n} |T_\lambda(f e^{-2\pi i \xi \cdot l})(y)|^r dy.$$

Next we can apply an assumption of $L^s - L^r$ boundedness of T_λ to see that

$$\|T_\lambda f\|_{W(L^\infty, \ell^r)}^r \leq C \|T_\lambda\|_{s \rightarrow r}^r \sum_l (1 + |l|)^{-N} \|f e^{-2\pi i \xi \cdot l}\|_s^r \leq C \|T_\lambda\|_{s \rightarrow r}^r \|f\|_s^r.$$

To get (2.2), we argue as in the proof of (1.1). Thus is, we first note that



$$\sum \|v^{-1/p} T_\lambda f\|_{L^{p'}(\mathbb{R}^n)}^{p'} \leq \sum \|V\|_{W(L^1, \ell^\infty)} \|T_\lambda f\|_{W(L^\infty, \ell^{p'})}^{p'}$$

By the preceding inequality with $r = p', s = q'$, this is majorized by

$$C \|V\|_{W(L^1, \ell^\infty)} \|T_\lambda\|_{q' \rightarrow p'}^{p'} \|f\|_{L^{q'}(\mathbb{R}^n)}^{p'}$$

Hence we obtain (2.2), assuming (2.4).

Proof of (2.4): Let $x, y \in Q$ and c_Q be the center of Q . Observe that

$$(T_\lambda f)(x) = a(x/\lambda) \int_{\mathbb{R}^m} \tilde{a}(x + c_Q) e^{i\lambda\phi(x/\lambda, \xi)\lambda - \phi(y/\lambda, \xi)} \tilde{b}(\xi) e^{i\lambda\phi(y/\lambda, \xi)} b(\xi) f(\xi) d\xi \quad (2.5)$$

where \tilde{a}, \tilde{b} are smooth functions satisfying $\tilde{a} = 1$ on $Q(0,1), \tilde{b}b = b$, and in addition they are assumed to be supported in $2Q(0,1)$ and $Q^m(0,1)$, respectively. Now we expand the function $\tilde{a}(x + c_Q) e^{i\lambda\phi(x/\lambda, \xi)\lambda - \phi(y/\lambda, \xi)} \tilde{b}(\xi)$ into a Fourier series on the box $2Q(0,1) \times Q^m(0,1)$ to get

$$\tilde{a}(x + c_Q) e^{i\lambda\phi(x/\lambda, \xi)\lambda - \phi(y/\lambda, \xi)} \tilde{b}(\xi) = \sum_{l,k} C_{l,k} e^{\pi i x \cdot k} e^{2\pi i \xi \cdot l} \quad (2.6)$$

By integration by parts, it is easy to see that for any $N > 0, |C_{k,l}| \leq C_N (1 + |l| + |k|)^{-N}$ with C_N independent of Q , because

$$\left| \partial_x^\alpha \partial_\xi^\beta [\lambda(\phi(x/\lambda, \xi)\lambda - \phi(y/\lambda, \xi))] \right| \leq C_{\alpha,\beta}$$

uniformly in λ, x, ξ . Finally, by plugging (2.6) into the integral (2.5) and assuming over k , we obtain (2.4).

3. Restriction estimates for curves with two weight functions

Let us recall the definition of the Fourier extension operator

$$(Tf)(x) = \int_I e^{2\pi i x \cdot \gamma(t)} f(t) dt, \quad x \in \mathbb{R}^n,$$

where I is an interval (not necessarily finite in this context), and $\gamma(t)$ is a nondegenerate curve in \mathbb{R}^n as before. Recall from (30) that if

$$\frac{1}{a} + \frac{(2+\epsilon)(3+\epsilon)}{2b} = 1, \quad b > [(2 + \epsilon)^2 + (4 + \epsilon)]/2, \text{ then}$$

$$\sum \|f\|_{L^b(\mathbb{R}^n)} \leq C \sum \|f\|_{L^a(\mathbb{R})} \quad (3.1)$$

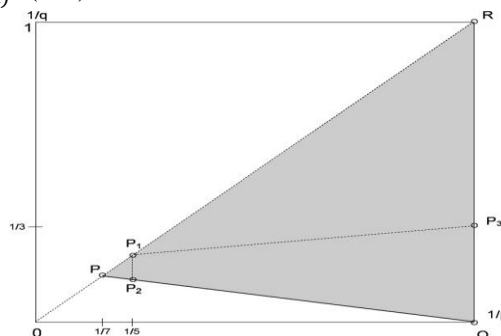


Fig. 2. The range of $(1/p, 1/q)$ in Theorem 3.1 for $n = 3$.

The next result extends the region of validity of a result of Lakey [11] to the full range of exponents (except possibly for the points on the upper edge (P, R) of the region Δ defined below). To be more precise, the conditions on p, q given in Theorem (3.1) state that $(1/p, 1/q)$ is any point in the region Δ , which is defined to be the union of the open triangle PQR (see Fig. 2) and the critical line segment (P, Q) , where;

$P = (2/((2 + \epsilon)^2 + (4 + \epsilon)), 2/(2 + \epsilon)^2 + (4 + \epsilon)), Q = (1,0)$ and $R = (1,1)$. The result may be regarded as a Fourier extension operator version of the elegant two-weight norm inequality for the Fourier transform, due to Muckenhoupt [12]. The proof is based on an



adaptation of his method. (The result in [11] is not formulated with the general hypothesis (3.2) below, but with more restrictive hypothesis (3.4) which only involves a single weight. It was proved in the open quadrilateral $p_1 p_2 Q p_3$, which is a proper subset of Δ . Here

$$\begin{aligned} p_1 &= (1/(4 + \epsilon), 1/(4 + \epsilon)), \\ p_2 &= (1/(4 + \epsilon), 2/[(2 + \epsilon)(4 + \epsilon)]), \text{ and} \\ p_3 &= (1, 1/(2 + \epsilon)). \end{aligned}$$

Corollary(3.1) [14]: Let $1 < p < q$, and $1/p + (2 + \epsilon)(3 + \epsilon)/(2q) \geq 1$. (Hence, $1/p > 2/((2 + \epsilon)^2 + (4 + \epsilon))$). Suppose that $U(x)$ and $V(t)$ are nonnegative functions on \mathbb{R}^n and \mathbb{R} , respectively. If there is a constant $A < \infty$ such that

$$\sum \sup_{s>0} \left(\int_0^s U^*(t) dt \right)^{1/q} \left(\int_0^{s^{\frac{-2}{(2+\epsilon)(3+\epsilon)}}} (V^{-p'/p})^*(t) dt \right)^{1/p} \leq A \quad (3.2)$$

Then

$$\sum \left(\int_{\mathbb{R}^n} |Tf(x)|^q U(x) dx \right)^{1/q} \leq CA \sum \left(\int_{\mathbb{R}} |f(t)|^p V(t) dt \right)^{1/p} \quad (3.3)$$

Here, $h^*(t), t \geq 0$, denotes the nonincreasing rearrangement of a function h on \mathbb{R}^n or \mathbb{R} , depending on the context.

If we take $U(x) = v(x)^{1-q}$ and $V(t) \equiv 1$ in (3.2), it (essentially) reduces to the following condition in [11]:

$$\sum \sup_{s \geq c} \left(s^{-\frac{2q}{(2+\epsilon)(3+\epsilon)p'}} \int_0^s (v^{1-q})^*(t) dt \right)^{1/q} \leq C \quad (3.4)$$

for some constants C and $c > 0$.

To prove Theorem (3.1), we will need the following two lemmas. To be more precise, they are only used to handle the cases $1/p + (2 + \epsilon)(3 + \epsilon)/(2q) > 1$, while we just need (3.1) when $1/p + (2 + \epsilon)(3 + \epsilon)/(2q) = 1$.

Lemma (3.2)[14]: Let $U(x)$ be a nonnegative function, and let $1 < p < q$, and $1/p + (2 + \epsilon)(3 + \epsilon)/(2q) > 1$. Put $\alpha = 2q/[(2 + \epsilon)((3 + \epsilon)p'] - 1$. If $U^*(t) \leq t^\alpha$, then we have

$$\left(\int_{\mathbb{R}^n} |Tf(x)|^q U(x) dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}} |f(t)|^p dt \right)^{1/p} \quad (3.5)$$

Proof: We may assume $U^*(t) = t^\alpha$. Note that $-1 < \alpha < 0$. So, the presence of the weight function $U^*(t)$ (or $U(t)$) is help full when $t > c > 0$, thus enabling the inequality (3.5) to hold, which would not hold for these p, q , in the absence of $U(t)$.

Given $(1/p, 1/q)$ as above, we may find a point $(1/a, 1/b)$ on the line segment $(1/a + (2 + \epsilon)(3 + \epsilon)/(2b) = 1, 0 < 1/b < 2/((2 + \epsilon)^2 + (4 + \epsilon))$, such that $(1/p, 1/q)$ is on the open line segment joining the points $(1, 1)$ and $(1/a, 1/b)$. A short calculation shows that

$$b = 1 + \frac{(2 + \epsilon)(3 + \epsilon)p'}{2q'} > q.$$

The conclusion by a weak-type interpolation. Consider a function $U_1(x)$ with $(1/U_1)^*(t) = t^{-1/(b-1)}$ for the above number b . We claim that

$$\sum \| (Tf) U_1 \|_{L^{1,\infty}(U_1^{-b} dx)} \leq C \sum \| f \|_1 \quad (3.6)$$

This may be seen as follows. We may clearly assume $\|f\|_1 > 0$. Since

$$\sum \| Tf \|_\infty \leq \sum \| f \|_1,$$

we have



$$\begin{aligned} \int_{|Tf|U_1>s} U_1^{-b} dx &\leq \int_{U_1>\frac{s}{\|f\|_1}} U_1^{-b} dx \\ &\leq \int_{(1/U_1)^*(t)<\frac{\|f\|_1}{s}} (U_1^{-b})^*(t) dt \\ &\leq \int_{t>(s/\|f\|_1)^{b-1}} t^{-\frac{b}{b-1}} dt \\ &= C \frac{\|f\|_1}{s} \end{aligned}$$

for $s > 0$. On the other hand we can rewrite (3.1) as

$$\sum \| (Tf)U_1 \|_{L^b(U_1^{-b} dx)} \leq C \sum \| f \|_a \quad (3.7)$$

Therefore, by applying the Marcinkiewicz interpolation theorem to (3.6) and (3.7), we obtain

$$\sum \left(\int |Tf|^q U_1^{q-b} dx \right)^{1/q} \leq C \sum \left(\int |f(t)|^p dt \right)^{1/p} \quad (3.8)$$

Let us now write $U = U_1^{q-b} = (1/U_1)^{b-q}$. Then the above condition on $U_1(x)$ corresponds to the condition $U^*(t) = t^\alpha$, for $\alpha = 2q/[(2 + \epsilon)(3 + \epsilon)p'] - 1$.

Lemma (3.3):(See [12]). Suppose that $U(x)$ is a nonnegative function on \mathbb{R}^n , and $|\{x: U(x) = t\}| = 0$ for all $t > 0$. Suppose that $S = |\{x: U(x) > 0\}| < \infty$, and let $L(t) = |\{x: U(x) \geq t\}|$ and $g(x) = L(U(x))$. Then $|\{x: g(x) < a\}| = \min(a, S)$ for $a > 0$.

Moreover, $(1/g)^*(t) = 1/t$, for $0 < t \leq S$, and $(1/g)^*(t) = 0$, for $t > S$.

Proof of corollary(3.1): Recall that $\alpha = 2q/[(2 + \epsilon)((3 + \epsilon)p')] - 1 \in (-1, 0]$. Let $\beta = \alpha + 1$ and $B = A^q$. Decompose \mathbb{R}^n to write $\mathbb{R}^n = \cup_{j \in \mathbb{Z}} E_j$, where $E_j = \{x: 2^{j\beta} Bg(x)^\alpha < U(x) \leq 2^{(j+1)\beta} Bg(x)^\alpha\}$ with $g(x)$ in Lemma (3.3). (When $\alpha = 0$, we have $g(x)^\alpha \equiv 1$ and Lemma (3.3) is not needed, and the argument given below simplifies somewhat).

If we also put $A_j = \{t: 2^{j\beta p/q} \leq 2V(t)\}$ it follows that

$$\int |Tf|^q U(x) dx \leq C_q \left(\sum_j \int_{E_j} |T(f\chi_{A_j})|^q U(x) dx + \sum_j \int_{E_j} |T(f\chi_{A_j^c})|^q U(x) dx \right).$$

Call the first and second sums in the parentheses J_1 and J_2 , respectively.

Let us first estimate J_1 . Since $U(x) \leq 2^{(j+1)\beta} Bg(x)^\alpha$ on E_j , we have

$$\begin{aligned} J_1 &\leq B \sum_j 2^{(j+1)\beta} \int |T(f\chi_{A_j})|^q g(x)^\alpha dx \\ &\leq BC \sum_j 2^{(j+1)\beta} \left(\int |f|^p \chi_{A_j} dt \right)^{q/p} \\ &\leq BC 2^\beta \left(\int |f|^p \sum_j 2^{j\beta p/q} \chi_{A_j}(t) dt \right)^{q/p} \\ &\leq BC_{p,q,\beta} \left(\int |f|^p V(t) dt \right)^{q/p}. \end{aligned}$$



If $\alpha < 0$, then $g(x)^\alpha$ has the decreasing rearrangement $(1/g)^*(t)^{-\alpha}$, which is $\leq t^\alpha$ by Lemma (3.3), and so the sectionond inequality above follows from Lemma (3.2) (and just from (3.1) when $\alpha = 0$). The third inequality follows from Minkowski's inequality, since $q/p \geq 1$. To show the last inequality, for each fixed t , let $j_0 = j_0(t)$ be the largest integer such that $2^{j_0\beta p/q} \leq 2V(t)$. Since $\beta > 0$, we have $\sum_{-\infty < j < \infty} 2^{j\beta p/q} \chi_{A_j}(t) \leq \sum_{-\infty < j < j_0} 2^{j\beta p/q} \leq CV(t)$.

Next let us estimate j_2 . We have that $2^j < Y(x) := B^{-1/\beta} U(x)^{1/\beta} g(x)^{-\alpha/\beta}$ on E_j . So

$$J_2 \leq \sum \int_{\mathbb{R}^n} \left(\int_{2V(t) < Y(x)^{\beta p/q}} |f(t)| dt \right)^q U(x) dx \quad (3.9)$$

Let $V_0(t)$ be a function such that $V(t) \leq 2V_0(t) \leq 2V(t)$ for all t , and $|\{t: V_0(t) = r\}| = 0$, for all $r > 0$. Let J be the smallest integer satisfying $2^J \geq \|f\|_1$. For $j < J$, choose a number $r_j > 0$ such that

$$\int_{2V_0(t) < r_j^{\beta p/q}} |f(t)| dt = 2^j,$$

and put $r_j = \infty$. Note that the definition of r_j gives the following equaity:

$$\int_{2V_0(t) < r_j^{\beta p/q}} |f(t)| dt = 2^j = 4 \int_{r_{j-2}^{\beta p/q} < 2V_0(t) < r_{j-1}^{\beta p/q}} |f(t)| dt \quad (3.10)$$

By a decomposition using r_j , we may bound the right-hand side of (3.9) by

$$\sum_{-\infty < j \leq J} \int_{r_{j-1} < Y(x) \leq r_j} \left(\int_{2V_0(t) < r_j^{\beta p/q}} |f(t)| dt \right)^q U(x) dx$$

By (3.10), this is bounded by 4^q times

$$\sum_{-\infty < j \leq J} \left(\int_{r_{j-1} < Y(x) \leq r_j} U(x) dx \right) \left(\int_{\leq 2V_0(t) < r_{j-1}^{\beta p/q}} |f(t)| dt \right)^q \quad (3.11)$$

From (3.2) we see that $|\{t: V(t) = 0\}| = 0$ and also that $f(t) = 0$ almost everywhere on the set where $(t) = \infty$. So, using the convention $0 \cdot \infty = 0$, we may write

$$\int_{r_{j-2}^{\beta p/q} \leq 2V_0(t) < r_{j-1}^{\beta p/q}} |f(t)| dt = \int_{r_{j-2}^{\beta p/q} \leq 2V_0(t) < r_{j-1}^{\beta p/q}} |f(t)| V(t)^{1/p} V(t)^{-1/p} dt.$$

By Hölder inequality and the fact that $q/p \geq 1$, (3.11) is bounded by

$$A_1 \sum_{-\infty < j \leq J} \left(\int_{\leq 2V_0(t) < r_{j-1}^{\beta p/q}} |f(t)|^p V(t) dt \right)^{q/p} \leq A_1 \left(\int |f|^p V \right)^{q/p}$$

where

$$A_1 := \sup_j \left(\int_{r_{j-1} < Y(x)} U(x) dx \right) \left(\int_{2V_0(t) < r_{j-1}^{\beta p/q}} V(t)^{-p'/p} dt \right)^{q/p'}$$

This yields



$$J_2 \leq A_1 \sum 4^q \left(\int |f(t)|^p V(t) dt \right)^{q/p}$$

completing the proof of the theorem, except that we still need to verify that $A_1 \leq A^q$.

We will now show that (3.2) implies that

$$A_1 \leq A^q \quad (3.12)$$

To prove this, let us first bound A_1 by an expression that does not involve the expression r_j . If $\alpha = 0$, then

$$\int_{r_{j-1} < Y(x)} U(x) dx = \int_{U(x) > Br_{j-1}} U(x) dx.$$

Since $V(t) \leq 2V_0(t)$, it follows that, when $\alpha = 0$ (so that $\beta = 1$), to show (3.12) it is enough to see that for every $r > 0$,

$$\sum \left(\int_{U(x) > Br} U(x) dx \right)^{1/q} \left(\int_{V(t) < r^{p/q}} V(t)^{-p'/p} dt \right)^{1/p'} \leq A \quad (3.13)$$

When $\alpha < 0$, let us first make the extra assumptions that the set $\{x: U(x) = r\} = \emptyset$ for each $r > 0$. (These assumptions will be removed later).

Put

$$E = \{y > 0: B^{-1/\beta} y^{1/\beta} L(y)^{-\alpha/\beta} > r_{j-1}\}.$$

Then

$$\int_{r_{j-1} < Y(x)} U(x) dx = \int_{U(x) \in E} U(x) dx = \int_{U^*(t) \in E} U^*(t) dt.$$

Since $L(U^*(t)) = t$ on the set where $U^*(t) > 0$ by Lemma (3.3), we have

$$\int_{U^*(t) \in E} U^*(t) dt = \int_{B^{-1/\beta} U^*(t)^{1/\beta} > r_j} U^*(t) dt.$$

Thus, to get (3.12) it is enough to show that for every $r > 0$,

$$\sum \left(\int_{B^{-1/\beta} U^*(t)^{1/\beta} t^{-\alpha/\beta} > r} U^*(t) dt \right)^{1/q} \left(\int_{V(t) < r^{\beta p/q}} V(t)^{-p'/p} dt \right)^{1/p'} \leq A \quad (3.14)$$

Note that (3.14) reduces to (3.13), when $\alpha = 0$ (and $\beta = 1$). Thus in both cases (i.e. $\alpha = 0$ and $\alpha < 0$) it only remains to show that (3.2) implies (3.14) (in fact these two conditions are equivalent -see [13]). Fix $r > 0$, and let s be such that

$$U^*(s)^{1/\beta} > B^{1/\beta} r s^{\alpha/\beta} \Leftrightarrow s U^*(s) > B r^\beta s^{\alpha+1}.$$

If there is no such s , the first integral of (3.14) is 0, and so (3.14) holds because of the convention $0 \cdot \infty = 0$. Since U^* is decreasing, we have

$$B r^\beta s^{\alpha/\beta} \leq \sum_0^s \int_0^s U^*(t) dt.$$

Let us set

$$W(t) = (V^{-p'/p})^*(t).$$

Since W is also decreasing, we get

$$\sum s^{-\beta p'/q} W(s^{-\beta p'/q}) \leq \sum_0^{s^{-\beta p'/q}} \int_0^{s^{-\beta p'/q}} W(t) dt.$$

Combining these two estimates, and using (3.2) and the fact that $B = A^q$, we obtain



$$\sum [Br^\beta s^{\alpha/\beta}]^{1/q} [s^{-\beta p'/q} W(s^{-\beta p'/q})]^{1/p'} \leq \sum \left(\int_0^s U^*(t) dt \right)^{1/q} \left(\int_0^{s^{-\beta p'/q}} W(t) dt \right)^{1/p'} \leq A.$$

Recall that $\beta = \alpha + 1$, and $\beta p'/q = 2/[(2 + \epsilon)(3 + \epsilon)]$. The last inequality thus gives

$$W(s^{-\beta p'/q}) \leq r^{-\beta p'/q}.$$

Again using the fact that W is decreasing, we obtain

$$\begin{aligned} \int_{V(t) < r^{-\beta p'/q}} V(t)^{-p'/p} dt &= \int_{V(t)^{-p'/p} > r^{-\beta p'/q}} V(t)^{-p'/p} dt \\ &= \int_{\substack{W(t) > r^{-\beta p'/q} \\ s^{-\beta p'/q}}} W(t) dt \\ &\leq \int_0^{s^{-\beta p'/q}} W(t) dt. \end{aligned}$$

Therefore,

$$\sum \left(\int_0^s U^*(t) dt \right)^{1/q} \left(\int_{V(t) < r^{-\beta p'/q}} V(t)^{-p'/p} dt \right)^{1/p'} \leq A \quad (3.15)$$

This holds also in the limiting case, i.e. for

$$s = \sup\{t: U^*(t)^{1/\beta} > B^{1/\beta} r t^{\alpha/\beta}\}.$$

For this value of s , we have

$$\sum_{B^{-1/\beta} U^*(t)^{1/\beta} t^{-\alpha/\beta} > r} \int U^*(t) dt \leq \sum_0^s \int U^*(t) dt.$$

Hence, the left-hand side of (3.14) is bounded by

$$\sum \left(\int_0^s U^*(t) dt \right)^{1/q} \left(\int_0^{s^{-\frac{2}{n(n+1)}}} V(t)^{-p'/p} dt \right)^{1/p'} \leq A.$$

The last inequality is true by (3.2). Therefore, we conclude that (3.2) implies (3.14), hence also (3.12).

Finally, in the case $\alpha < 0$, we need to remove the extra assumptions that the set $\{x: U(x) > 0\}$ has finite measure and that $|\{x: U(x) = r\}| = 0$ for all $r > 0$. First, if the condition that $|\{x: U(x) = r\}| = 0$ fails for some $r > 0$, then we may just replace $U(x)$ by some function $U_0(x)$ such that $U(x) \leq 2U_0(x) \leq 2U(x)$ for all x and $|\{x: U_0(x) = r\}| = 0$ for all $r > 0$. If we can prove (3.3) for $U_0(x)$, then it obviously holds also for U . Now suppose that $|\{x: U(x) > s\}| = \infty$ for some $s > 0$. Then for $\alpha < 0$ the set where $B^{1/\beta} U^*(t)^{1/\beta} t^{-\alpha/\beta} > r$ ($\Leftrightarrow U^*(t) > Br^\beta t^\alpha$) and $U^*(t) > s$ has infinite measure. Thus, the first integral in (3.14) is infinite for all r , so $V(x) = \infty$ almost everywhere and (3.14) holds trivially. So we may assume that

$|\{x: U(x) > s\}| < \infty$ for all $s > 0$. If we define a sequence of functions

$$U_k(x) = U(x) \cdot \chi_{\{U > 1/k\}}(x)$$

then it follows that $|\{x: U_k(x) > 0\}| = |\{x: U(x) > 1/k\}| < \infty$. Hence, by the argument just given, (3.3) is satisfied for U_k in place U . The monotone convergence theorem now gives (3.3) for the more general weight function U .



References

- G. Arkhipov, A. Karachuba, V. Chubarikov, Exponent of convergence of the singular integral in the Tarry problem, Dokl. Akad. Nauk SSSR 248 (1979) 268–272; English translation: Soviet Math. Dokl. 20 (1980) 978–981.
- G. Arkhipov, A. Karachuba, V. Chubarikov, Trigonometric Sums in Number Theory and Analysis, de Gruyter Exp. Math., vol. 39, de Gruyter, Berlin, 2004. Translated from the 1987 Russian original.
- J.-G. Bak, S.H. Lee, Estimates for an oscillatory integral operator related to restriction to space curves, Proc. Amer. Math. Soc. 132 (2004) 1393–1401.
- J.-G. Bak, D.M. Oberlin, A note on Fourier restriction for curves in \mathbb{R}^3 , in: Proceedings of the AMS Conference on Harmonic Analysis, Mt Holyoke College, June 2001, in: Contemp. Math., vol. 320, 2003, pp. 9–13.
- S. Drury, Restriction of Fourier transforms to curves, Ann. Inst. Fourier (Grenoble) 35 (1985) 117–123.
- S. Drury, Degenerate curves and harmonic analysis, Math. Proc. Cambridge Philos. Soc. 108 (1990) 89–96.
- S. Drury, B. Marshall, Fourier restriction theorems for curves with affine and Euclidean arclengths, Math. Proc. Cambridge Philos. Soc. 97 (1985) 111–125.
- S. Drury, B. Marshall, Fourier restriction theorems for degenerate curves, Math. Proc. Cambridge Philos. Soc. 101 (1987) 541–553.
- J. Fournier, J. Stewart, Amalgams of L^p and ℓ^q , Bull. Amer. Math. Soc. (N.S.) 13 (1) (1985) 1–21.
- F. Holland, Harmonic analysis on amalgams of L^p and ℓ^q , J. London Math. Soc. (2) 10 (1975) 295–305.
- J. Lakey, Weighted restriction for curves, Canad. Math. Bull. 36 (1993) 87–95.
- B. Muckenhoupt, Weighted norm inequalities for the Fourier transform, Trans. Amer. Math. Soc. 276 (1983) 729–742.
- B. Muckenhoupt, A note on two weight function conditions for a Fourier transform norm inequality, Proc. Amer. Math. Soc. 101 (1987) 541–553.
- Jong-Guk Bak, Jungjin Lee, Sanghyuk Lee, Weighted restriction theorems for space curves, J. Math. Anal. Appl. 334 (2007) 1232–1245.